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The incidence coloring conjecture for graphs of maximum degree 3

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Abstract

The incidence coloring conjecture, or ICC, states that any graph can be incidence-colored with $\Delta+2$ colors, where Δ is the maximum degree of the graph. After being introduced in 1993 by Brualdi and Massey, ICC was shown to be false in general by Guiduli in 1997, following the work of Algor and Alon. However, Shiu, Lam and Chen conjectured that the ICC holds for cubic graphs and proved it for some classes of such graphs. In this paper we prove the ICC for any graph with $\Delta = 3$.

Keywords: Cubic graph; Incidence coloring

1. Introduction

Unless specifically noted, all the graphs in this paper are finite, undirected, and simple. Let $G=(V,E)$ be a graph. Let

$$I(G)=\{(v,e): v\in V, e\in E, \text{ and } v \text{ is incident with } e\}$$

be the set of all *incidence pairs* of G.

Following Shiu et al. [6] we view G as a digraph by splitting each edge uv into two opposite arcs (u,v) and (v,u). For $e=uv$, we identify the incidence pair (u,e) with the arc (u,v). By a slight abuse of notation we will refer to the incidence pair (u,v) whenever it is convenient to do so.

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Two (distinct) incidence pairs (v, e) and (w, f) are *adjacent* if at least one of the following holds:

(i) $v = w$,
(ii) $e = f$,
(iii) $e = vw$, or
(iv) $f = vw$.

We define an *incidence coloring* σ of G to be a map from $I(G)$ to the *color set* S such that adjacent incidence pairs are assigned different colors. If $\sigma : I(G) \to S$ is an incidence coloring with $|S| = k$, then we say that σ is a *k-incidence coloring of G*. The *incidence chromatic number* of G, denoted $\chi_i(G)$, is the smallest k for which there exists a k-incidence coloring of G.

In [2], Brualdi and Massey developed the concept of incidence coloring and made the incidence coloring conjecture, or ICC, which states that $\chi_i(G) \leqslant \Delta(G)+2$, where $\Delta = \Delta(G)$ is the maximum degree of a vertex of G. In [5], Guiduli observed that incidence coloring is a special case of directed star arboricity, introduced by Algor and Alon in [1]. Based on results in [1], Guiduli showed that $\chi_i(G) \geqslant \Delta + \Omega(\log \Delta)$, thus demonstrating the ICC to be false. However, this asymptotic result left open the possibility that the ICC is true for graphs with small maximum degree Δ. Cases $\Delta = 1$ and $\Delta = 2$ are immediate. In [6], Shiu et al. proved that $\chi_i(G) \leqslant 5$ for several classes of cubic (3-regular) 2-connected graphs G, including Hamiltonian cubic graphs, and conjectured that same holds for arbitrary cubic graphs.

In this paper, we show that $\chi_i(G) \leqslant 5$ for all graphs G with $\Delta(G) = 3$. In what follows, “color” and “coloring” will mean “incidence color” and “incidence coloring”. Our color set is $\{1, 2, 3, 4, 5\}$ throughout.

In Section 2, we obtain two technical results. Proposition 3 proves that all graphs of a certain type can be decomposed into a sum of a 1-factor and a 2-regular graph. Proposition 4 describes a 5-coloring of graphs with such a decomposition. In Section 3, we apply these results to prove that cubic graphs are 5-colorable. The 5-colorability of any graph G with $\Delta(G) = 3$ follows as a corollary at the end of Section 3.

The following construction will appear several times. Given a cycle $C = x_0 x_1 \ldots x_{p-1} x_0$ we define the *default cycle coloring from x_0 to x_{p-1}* as follows:

$$\sigma(x_k, x_{k+1}) = \begin{cases} 1 & : \quad k \equiv 0 \pmod 3, \\ 2 & : \quad k \equiv 1 \pmod 3, \\ 3 & : \quad k \equiv 2 \pmod 3, \end{cases}$$

and

$$\sigma(x_{k+1}, x_k) = \begin{cases} 3 & : \quad k \equiv 0 \bmod 3, \\ 1 & : \quad k \equiv 1 \bmod 3, \\ 2 & : \quad k \equiv 2 \bmod 3, \end{cases}$$

for all k from 0 to $p-1$ (see Fig. 1). (We interpret x_p as x_0.)

Note that this assigns colors to all incidence pairs of C, but, in general, does NOT produce an incidence coloring of C because adjacent incidences at the vertex x_0 may be colored the same color.

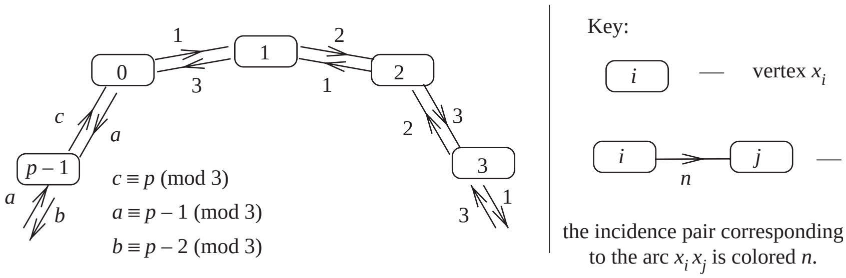

Fig. 1. Default cycle coloring.

2. Technical propositions

A connected component of a graph is *odd* if it has an odd number of vertices. A graph is *Δ-regular* if each vertex has degree Δ. A *1-factor*, or a perfect matching, of G is a spanning 1-regular subgraph of G.

Let $q(H)$ be the number of odd components of the graph H. We will need the following results.

Lemma 1 (*Tutte's condition*). *A graph G has a* 1-*factor if and only if* $q(G - S) \leqslant |S|$ *for all* $S \subseteq V(G)$.

Lemma 2 (*Petersen's Theorem*). *Suppose G is a* 2-*connected cubic graph. Then G is the sum of a* 1-*factor and a* 2-*regular graph.*

The proofs of both of the above statements can be found in [4, p. 35]. We use Lemma 1 to prove the following minor modification of Petersen's Theorem.

Proposition 3. *Suppose that* v *is a degree* 1 *vertex of a graph G and that all other vertices of G are of degree* 3. *Further suppose that* $G - v$ *is* 2-*edge connected. Then G is a sum of a* 1-*factor and a* 2-*regular graph.*

Proof. Observe that it is sufficient to show that G has a 1-factor. We will show that G satisfies Tutte's condition for the existence of a 1-factor; this will suffice by Lemma 1.

Let $S \subseteq V(G)$ be given and consider an odd component C of $G - S$. Since all vertices of G have odd degree, the degrees in C sum to an odd number, but only an even part of this sum arises from edges of C. So G has an odd number of edges between S and C. Therefore either the only such edge is a bridge, or G has at least 3 such edges. Since there is only one bridge (the edge incident with v), the total number of edges between S and $G - S$ is at least $3q(G - S) - 2$. But as each vertex in S has degree no greater than 3, it is also at most $3|S|$. So $3q(G - S) - 2 \leqslant 3|S|$ and $q(G - S) \leqslant |S|$ as required. □

We now prove that the graphs in a certain class containing all the graphs satisfying the hypotheses of Proposition 3 are 5-colorable.

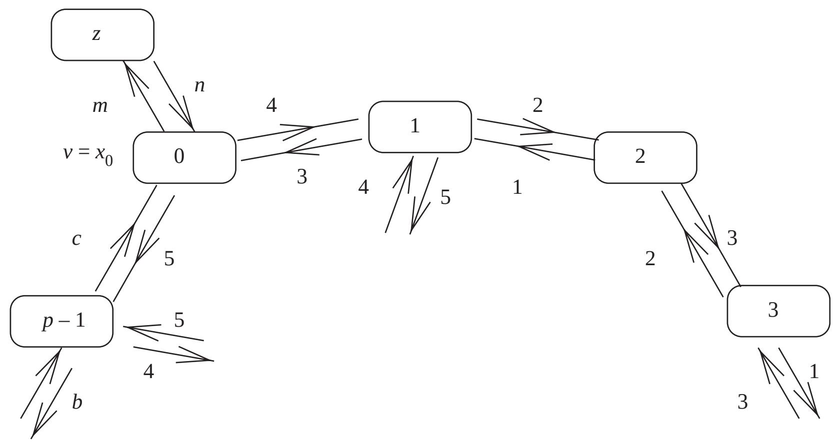

Fig. 2. Coloring C.

Proposition 4. *Let* $G = C + F$, *where* C *is a* 2*-regular graph and* F *is a* 1*-factor*, *and further suppose that* G *has no more than* 1 *vertex of degree* 1 (*which means that* C *spans* G *with the exception of the possible single vertex of degree* 1). *Then* G *is* 5*-colorable*.

Proof. Note that C is a disjoint union of cycles $C_0, \ldots, C_\ell$. First consider the case when $\ell = 0$ and $C = C_0 = x_0 x_1 \cdots x_{p-1} x_0$. If G has no vertices of degree 1, then G is a Hamiltonian cubic graph, and is 5-colorable by [6], Theorem 2.4. If G has a degree 1 vertex z matched by F to v, then we renumber the vertices so that $v = x_0$ and default cycle color C. We then recolor by assigning $\sigma(x_0, x_1) = 4$ and $\sigma(x_0, x_{p-1}) = 5$ to get a proper coloring. If u and w are vertices matched to x_1 and x_{p-1}, respectively, by F, we assign $\sigma(u, x_1) = 4$, $\sigma(x_1, u) = 5$, and $\sigma(w, x_{p-1}) = 5$, $\sigma(x_{p-1}, w) = 4$ (note that this allows for the possibility that $w = x_1$ and $u = x_{p-1}$). For any remaining edge $e = xy$ of F with neither x nor y equal to any of x_0, x_1 or x_{p-1} we assign $\sigma(x, y) = 4$ and $\sigma(y, x) = 5$. The only incidences of C left uncolored are (z, v) and (v, z). Let one of the colors 1, 2, and 3 not used by $\sigma(x_1, v)$ and $\sigma(x_{p-1}, v)$ be m. Let n be a color from 1, 2, 3 different from m. We put $\sigma(v, z) = m$ and $\sigma(z, v) = n$ to complete a proper coloring of C (see Fig. 2).

Now consider the case $\ell > 0$. Define the *cycle connection multigraph* M by letting

$$V(M) = \{C_0, \ldots, C_\ell\}$$

and

$$E(M) = \{e \in E(F) : \text{ the vertices of } e \text{ are in } C_i \text{ and } C_j, \text{ with } i \neq j\}$$

with the map $E \mapsto V \cup [V]^2 : e \mapsto (C_i, C_j)$ (we use the definition of multigraph from [4, p. 24]). Note that M is loop-free. We let $T = (V(T), E(T))$ be a spanning tree of M. Renumber the C_i's so that C_{i+1} has exactly one neighbor in T among $C_1, \ldots, C_i$. Note that $V(T) = V(M)$ and $E(T) \subseteq E(M) \subset E(G)$. Call $e \in E(G)$ a *connection edge* if $e \in E(T)$. Call $v \in V(G)$ a *connection vertex* if it is incident with a connection edge.

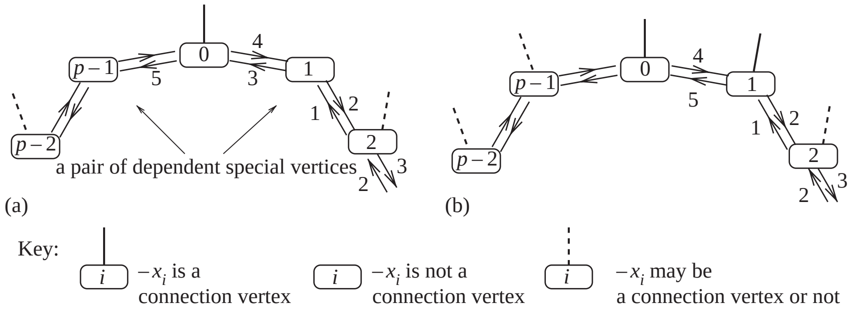

Fig. 3. Coloring x_0.

We wish to 5-color each C_i nicely, put them together, and then color the rest of the edges to get a 5-coloring of G.

Suppose the length of C_i is p and $C_i = x_0x_1 \cdots x_{p-1}x_0$. We make the identification $x_p = x_0$. Now we 5-color the incidence pairs of C_i.

If there exists a pair of connection vertices x_j and x_{j+1} in C_i, pick x_j as a starting vertex. If there are many such pairs, pick x_j from any of them. If there are no such pairs of connection vertices, pick an arbitrary connection vertex x_j in C_i to be a starting vertex. We cyclically renumber the vertices of C_i so that $j = 0$. Now we default cycle color C_i. As noted before, this assigns colors to all incidence pairs of C_i, but does not necessarily produce an incidence coloring of C_i because adjacent incidences at the vertex x_0 may be colored the same color. We now proceed to recolor some of the incidences to get a coloring of C_i, as follows:

(i) If x_1 is not a connection vertex, then assign $\sigma(x_0, x_{p-1}) = 5$ and $\sigma(x_0, x_1) = 4$; mark x_1 and x_{p-1} as *special vertices* of G forming a *dependent pair* (see Fig. 3a). If x_1 is a connection vertex, then assign $\sigma(x_1, x_0) = 5$ and $\sigma(x_0, x_1) = 4$ (see Fig. 3b). Note that after this recoloring C_i is 5-colored.

(ii) Go to the next connection vertex (i.e. the connection vertex x_j with smallest j not involved in any of the previous steps). If x_{j+1} is a connection vertex and is not x_0, assign $\sigma(x_{j+1}, x_j) = 5$ and $\sigma(x_j, x_{j+1}) = 4$ (see Fig. 4a). If x_{j+1} is not a connection vertex or $x_{j+1} = x_0$ then assign $\sigma(x_j, x_{j+1}) = 5$ (see Figs. 4b–d). Note that by our choice of the starting vertex, if x_{p-1} is a connection vertex, then x_1 is also a connection vertex.

(iii) Repeat (ii) until no more connection vertices are left.

This completes the coloring of the C_i's.

We now proceed to construct a coloring of G in several steps. For the first step we will need the following lemma.

Lemma 5. *Suppose u_0 and w_0 are degree 2 vertices of the graphs U and W respectively. Let u_{-1}, u_1 be the vertices adjacent to u in U and let w_{-1} and w_1 be the vertices joined to*

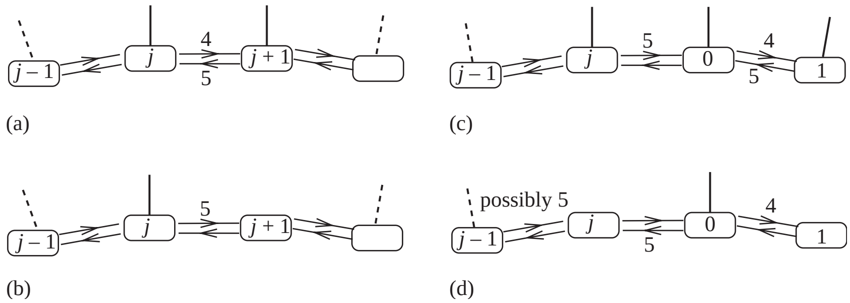

Fig. 4. Coloring x_j.

w_0 in W. Suppose that U and W are 5*-colored by maps σ and ρ respectively* (*see Fig.* 5a). *Suppose further that among the colors of the two incidences* (u_0, u_i) *there is at most one of* 1, 2, *or* 3; *that among the colors of the four incidences* (u_i, u_j) (*where* $ij = 0$) *there are at most two of* 1, 2, *or* 3; *and that the same holds for the w_i's. Then if we join u_0 and w_0 by an edge $e = u_0w_0$, there is a permutation α of* $\{1, 2, 3\}$ *and a map* $\tau : \{(u_0, w_0), (w_0, u_0)\} \mapsto \{1, 2, 3\}$ *such that the maps σ, τ, and $\alpha \circ \rho$ together define a coloring of the combined graph* $G = (V(U) \cup V(W), E(U) \cup E(W) \cup \{uw\})$.

Proof of Lemma 5. There is at least one color in $\{1, 2, 3\}$ which is not any of $\sigma(u_i, u_j)$ (where $ij = 0$); call it a_u. There is at least one color in $\{1, 2, 3\}$ which is not any of $\sigma(u_0, u_j)$ and is not a_u; call it b_u. Finally, let c_u be the sole remaining color in $\{1, 2, 3\}$. Do the same with w. The permutation $\alpha : a_w \mapsto b_u, b_w \mapsto a_u, c_w \mapsto c_u$ and the map $\tau : (u_0, w_0) \mapsto a_u, (w_0, u_0) \mapsto b_u$ are as desired (see Figs. 5b and c). □

Using the spanning tree T of the connection multigraph M we define the subgraphs G_k of G by

$$V(G_k) = \bigcup_{i=0}^{k} V(C_i),$$

$$E(G_k) = \left(\bigcup_{i=0}^{k} E(C_i)\right) \cup \{\text{edges of } G \text{ corresponding to } (C_i, C_j) \in E(T) \text{ with } i, j \leqslant k\},$$

for k from 1 to ℓ. We now use induction on k to 5-color each G_k. The base case is proved by observing that the graphs C_0 and C_1 satisfy the conditions of Lemma 5, with the two vertices u_0 and w_0 being the vertices of the edge corresponding to the edge (C_0, C_1). Hence we can get a coloring of their union together with the edge connecting them, i.e. a coloring of G_1. For the induction step, observe that G_k and C_{k+1} satisfy the hypothesis of Lemma 5, so that we can color the sole edge of G between them, producing the coloring of G_{k+1}. This completes the induction and produces a 5-coloring of G_ℓ, which has all the edges of T in it.

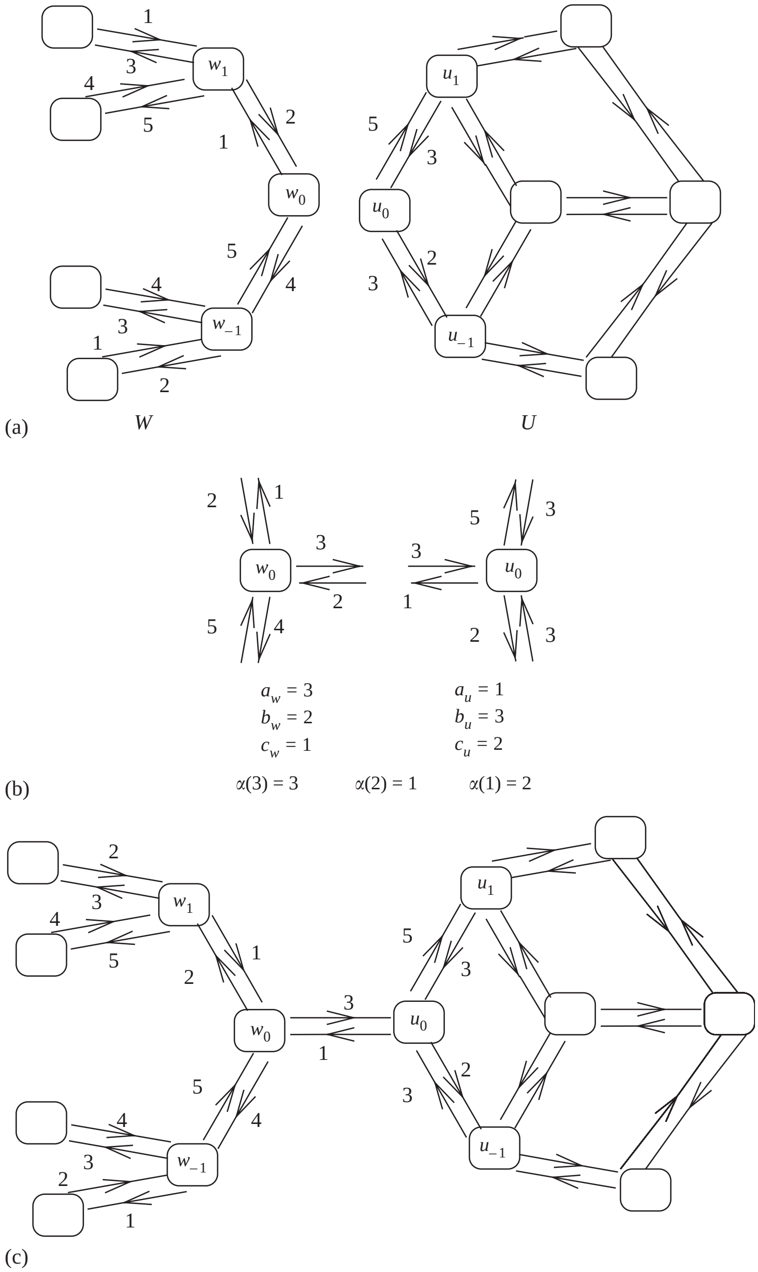

Fig. 5. Lemma 5 figures.

We now color the incidences involving the edges (of the 1-factor F) incident with special vertices. We start with a special vertex b_1 such that the other vertex (which we will call a_0) of the edge of F incident with b_1 is not a special vertex. If no such special vertex exists, we start with an arbitrary special vertex b_1. The vertex b_1 is in a dependence pair with another special vertex, which we call a_1. Starting from b_1 and a_1, we proceed to construct a sequence of special vertices. At each step, having constructed a_i, we denote by b_{i+1} the other vertex of the edge of F incident with a_i. There are three possibilities:

(i) Vertex b_{i+1} is special and different from b_1. We designate its paired dependent special vertex a_{i+1}, and proceed.
(ii) Vertex b_{i+1} is not special. We terminate the construction. Note that in this case $a_i b_{i+1}$ is an edge of F with one end point special and one not, and so a_0 from the edge $a_0 b_1$ must not be a special vertex either, for otherwise we would have chosen a_i instead of b_1 at the beginning. In this case we get a sequence of special vertices $b_1, a_1, b_2, \ldots, b_i, a_i$ with two non-special, non-connection vertices a_0 and b_{i+1} at the beginning and at the end, with a_j and b_j dependent for each j.
(iii) We have $b_{i+1} = b_1$. We terminate the construction. In this case we get a cyclic sequence of special vertices $b_1 = b_{i+1}, a_1, b_2 \ldots, b_i, a_i = a_0, b_{i+1}$, with a_j and b_j dependent for each j.

Now, we color $\sigma(b_{k+1}, a_k) = 4$ and $\sigma(a_k, b_{k+1}) = 5$ (for all k from 0 to i). This does not always produce a proper coloring, so we must recolor some incidence pairs. If any edge $e = a_k v$ of C has $\sigma(v, a_k) = 5$, we recolor it to $\sigma(v, a_k) = 4$, and if any edge $e = b_{k+1} v$ of C has $\sigma(v, b_{k+1}) = 4$, we recolor it to $\sigma(v, b_{k+1}) = 5$ (for all k from 0 to i). After this we do have a proper coloring.

We repeat the process, each time ignoring all previously used special vertices for which all incidences are already colored, until no more special vertices are left. This produces a proper coloring of C and all the edges of F which are either in T or have a special vertex.

For each edge uv such that u and v are not connection vertices and not special vertices we color $\sigma(u, v) = 4$ and $\sigma(v, u) = 5$. We also recolor any edge wv (where $w \neq u$) with $\sigma(w, v) = 5$ into $\sigma(w, v) = 4$ and any edge wu (where $w \neq v$) with $\sigma(w, u) = 4$ into $\sigma(w, u) = 5$. We now have a proper coloring of all incidence pairs of G and thus we are done. □

3. 5-coloring a cubic graph

We are now ready to describe a 5-coloring of an arbitrary cubic graph G.

Theorem 6. *Let $G = (V, E)$ be a cubic graph. Then G is 5-colorable.*

Proof. Let B be the set of bridges of $G = (V, E)$. Define $\widehat{G} = (V, E \backslash B)$ (i.e. G with all the bridges removed) and let $K = \{K_0, K_1, \ldots K_n\}$ be the set of all connected components of $\widehat{G}$ which are not single points. Denote the set of single point connected components of $\widehat{G}$

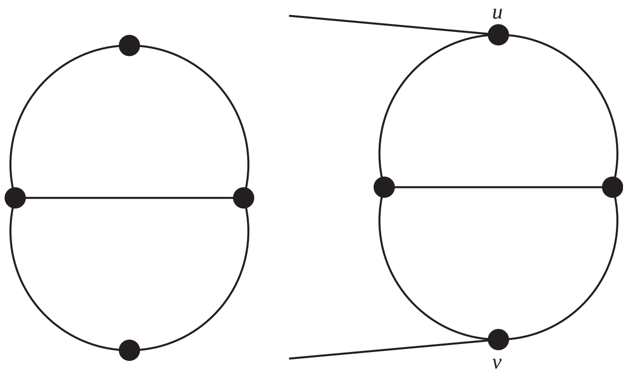

Fig. 6. Graph Θ.

as $D = \{D_0, D_1, \ldots, D_m\}$. Observe that each K_i is 2-edge connected, as any bridge of K_i would be a bridge of G.

For a subgraph $H = (V(H), E(H))$ of G, define the *completion of H in G with respect to B* to be the subgraph J of G with $E(J) = E(H) \cup \{e \in B :$ there exists $v \in V(H)$ incident with $e\}$ and $V(J) = \{v \in V(G) :$ there exists $e \in E(J)$ incident with $v\}$. Informally, J is H with all bridges of G incident with H attached. Let P_i be the completion of K_i in G with respect to B. Let Y_i be the completion of D_i in G with respect to B.

For an arbitrary vertex v of P_i, either v is a vertex of K_i as well, in which case v has degree 3 in P_i, or v is not a vertex of K_i. In this case v is a vertex of one of the attached bridges. Then v is a vertex of exactly one bridge in B_i, and therefore has degree 1 in P_i. Hence all vertices of P_i have degrees 1 or 3.

We now construct a graph P_i^* containing P_i in the following manner: we divide all degree 1 vertices of P_i, except possibly for one, into pairs, and attach a graph Θ (see Fig. 6) to each pair. Let P_i^* be the union of P_i with all the Θ's. All vertices of P_i^* are of degree 3, with possible exception of a single vertex v. Either P_i^* or $P_i^* - v$ is 2-edge connected. By combining Proposition 4 with either Lemma 2 or Proposition 3, the graph P_i^* is 5-colorable. Hence P_i is 5-colorable. Since each vertex of G has degree 3, each Y_j is a 3-star (having one vertex of degree 3 and three vertices of degree 1) and so is trivially 5-colorable (and even 4-colorable). We now make the following observation:

$$\text{All } P_i\text{'s and } Y_j\text{'s are 5-colorable.} \tag{1}$$

It remains only to produce a coloring of G from the colorings of each P_i and Y_j. In order to do that we consider $\widehat{G} = (V, E \backslash B)$ once again. Recall that connected components of $\widehat{G}$ other than points are in $K = \{K_0, K_1, \ldots, K_n\}$ and the single point components of $\widehat{G}$ are in $D = \{D_0, D_1, \ldots, D_m\}$. Define a *component connection graph N* by taking

$$V(N) = \{K_0, K_1, \ldots K_n, D_0, D_1 \ldots D_m\}$$

and

$$E(N) = \{(g_1, g_2) \in V(N) \times V(N) : \text{ there exist } w_1 \in V(g_1) \text{ and } w_2 \in V(g_2) \text{ such that } (w_1, w_2) \in E(G)\}.$$

Note that the edge-set $E(N)$ is in one-to-one correspondence with B. Indeed, each edge in B has end points in two different components of $\widehat{G}$, and so there is an edge of N corresponding to it. Conversely, any edge of G connecting components of $\widehat{G}$ is a bridge, and there is no more than one such edge connecting any pair of components.

Observe that N is a tree. Indeed, if there were a cycle in N, then by taking corresponding edges of B in G and connecting every consecutive pair of obtained edges by a path in G (which is always possible since each K_i is connected and each D_i is a single vertex), we obtain a cycle in G passing through bridges, which is a contradiction.

Now, we are in a situation similar to that of the last part of the proof of Proposition 4. We number the vertices of N by $g_0, \dots, g_d$ in such a way that g_{i+1} has exactly one neighbor among $g_0, \dots, g_i$. We will call an incidence pair (v, e) *incident* with a subgraph H of G if v is a vertex of H or if e is incident with (a vertex of) H.

Using induction on k, we will produce a 5-coloring of all incidence pairs of G incident in G with $g_0, \dots, g_k$. The base case is proved by observing that the set of all incidence pairs of G incident in G with g_0 is the set of all incidence pairs of P_i if $g_0 = K_i$ or of Y_j if $g_0 = D_j$, and so is 5-colorable by the observation (1). For the inductive step, observe that by the same logic, the set U of all incidence pairs of G incident in G with g_{k+1} is 5-colorable. By the induction hypothesis, the set W of all incidence pairs of G incident in G with $g_0, \dots, g_k$ is 5-colorable. Because g_{k+1} has exactly one neighbor among $g_0, \dots, g_k$, the sets U and W have exactly two incidence pairs (corresponding to the sole edge between g_0 and a previous vertex) in common. Renaming the colors in the coloring of U so as to make the colors of the two common incidence pairs in U the same as in W, we get a coloring of all incidence pairs of G incident in G with $g_0, \dots, g_k, g_{k+1}$. This completes the induction and produces a 5-coloring of G. $\square$

Since an arbitrary graph G with $\Delta(G) = 3$ can be embedded in a cubic graph, Theorem 6 implies all graphs of maximum degree 3 are 5-incidence colorable.

4. Future work

Resolving the ICC for graphs with $\Delta = 4$ would be an interesting direction for further research. A computer search confirmed that all 4-regular graphs with less than 10 vertices are 6-colorable. Overall, finding 6-colorings for 4-regular graphs seems to become easier as the number of vertices increases, which would indicate that the ICC may be true for graphs with $\Delta = 4$ in general.

We should note that all known counterexamples to the ICC come from the following lower bound for $\chi_i(G)$, arising from Lemma 3.1 in [1]:

$$\chi_i(G) \geqslant \frac{2|E(G)|}{|G| - \gamma(G)}, \tag{2}$$

where $\gamma(G)$, the domination number of G, is the minimum size of a set $D \subseteq V(G)$ so that for any $v \in V(G)$ either $v \in D$ or v is joined by an edge to $w \in D$.

For 4-regular graphs, (2) reduces to

$$\gamma(G) \geqslant \frac{\chi_i(G) - 4}{\chi_i(G)}|G|.$$

Therefore, if the ICC is true for 4-regular graphs, then $\gamma(G) \geqslant |G|/3$, which was proved by Clark and Dunning in [3, p. 19] for graphs of small order, but is an open question in general. We also note that the results of [3] combined with (2) provide counterexamples to the ICC with $\Delta = 6, 7$ and 8, and future computations of domination numbers in regular graphs of small order are likely to provide a counterexample with $\Delta = 5$. Somewhat surprisingly, the author has not encountered any regular graph with $\chi_i(G)$ bigger then the lower bound given by (2). It would be interesting to find such a graph or to prove that no such graph exists.

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